

On Concircular Transformations in Finsler Geometry

Zhongmin Shen and Guojun Yang*

Abstract

A geodesic circle in Finsler geometry is a natural extension of that in a Euclidean space. In this paper, we study geodesic circles and (infinitesimal) concircular transformations on a Finsler manifold. We characterize a concircular vector field with some PDEs on the tangent bundle, and then we obtain respectively necessary and sufficient conditions for a concircular vector field to be conformal and a conformal vector field to be concircular. We also show conditions for two conformally related Finsler metrics to be concircular, and obtain some invariant curvature properties under conformal and concircular transformations.

Keywords: Geodesic circle, Conformal/Concircular transformation, Flag curvature, Einstein metric, Lie derivative, Cartan Y -connection

MR(2000) subject classification: 53B40, 53C60

1 Introduction

A geodesic circle in a Euclidean space is a straight line or a circle with finite positive radius, and it can be generalized naturally to Riemann geometry by using Levi-Civita connection ([9]), or more generally generalized to Finsler geometry by the so called Cartan Y -connection introduced by M. Matsumoto in [1] [8] (also see Section 2 below). A curve $\gamma = \gamma(s)$ on a Finsler manifold (M, F) with s being the arc-length is called a geodesic circle if it satisfies

$$D_{\dot{\gamma}}^* D_{\dot{\gamma}}^* \dot{\gamma} + g_{\dot{\gamma}}(D_{\dot{\gamma}}^* \dot{\gamma}, D_{\dot{\gamma}}^* \dot{\gamma}) \dot{\gamma} = 0, \quad (1)$$

where D^* is the Cartan Y -connection (induced by $\dot{\gamma}$) and $g_{\dot{\gamma}}$ is the inner product induced by F . For two Finsler manifolds (M, F) and $(\widetilde{M}, \widetilde{F})$, a diffeomorphism φ from (M, F) to $(\widetilde{M}, \widetilde{F})$ is said to be concircular if φ maps geodesic circles to geodesic circles. For convenience, we say two Finsler metrics on a same manifold are concircular if they have the same geodesic circles as points set. Correspondingly, a vector field V on a Finsler manifold (M, F) is said to be concircular if its flow induces infinitesimal concircular transformations.

A Finsler metric $F = F(x, y)$ with $x \in M, y \in T_x M$ defines its fundamental metric tensor g_{ij} (while g^{ij} the inverse), Cartan torsion C_{jk}^i and mean Cartan torsion I_i respectively by

$$g_{ij} := \dot{\partial}_i \dot{\partial}_j (F^2/2), \quad 2C_{ijk} := \dot{\partial}_k g_{ij}, \quad I_i := g^{jk} C_{ijk} = C_{ir}^r, \quad (\dot{\partial}_i := \partial/\partial y^i).$$

It is well known that a Finsler metric is Riemannian iff. the Cartan torsion, or the mean Cartan torsion vanishes ([3]). In a Minkowski Finsler space, the geodesic circle equation (1) is reduced to a simple equation which is closely related to the Cartan torsion (Example 3.5 below). A vector field V on a manifold M induces a flow φ_t acting on M , and φ_t is naturally lifted to a flow $\widetilde{\varphi}_t$ on the tangent bundle TM , where $\widetilde{\varphi}_t : TM \mapsto TM$ is defined by

*Corresponding author, supported by the National Natural Science Foundation of China (11471226)

$\tilde{\varphi}_t(x, y) := (\varphi_t(x), \varphi_{t*}(y))$. Taking the derivative of $\tilde{\varphi}_t$ with respect to t at $t = 0$, we obtain a vector field V^c on the tangent bundle TM , which is called the complete lift of V . A vector field V on a Finsler manifold (M, F) is said to be conformal if F keeps conformally related under the flow $\tilde{\varphi}_t$, that is, it holds $\tilde{\varphi}_t^* F = e^{\sigma_t} F$, where σ_t is a function on M for every t , and then by taking the derivative of σ_t at $t = 0$ we obtain a scalar function ρ (on M) called a conformal factor. For some studies on conformal vector fields, one may refer to [12] [20] [21], for instance.

In 1940s, Yano introduced concircular transformations of Riemannian manifolds and developed the theory of concircular geometry in a series of papers ([11]). After that, some researchers did further jobs on concircular transformations in Riemann geometry (see for instance [5] [7] [10] [15]). Vogel shows that a concircular transformation of Riemannian manifolds is a conformal transformation ([15]), and Ishihara proves that a concircular vector field on a Riemannian manifold is a conformal vector field ([5]).

For Finsler manifolds, one is wondering whether a concircular transformation, or a concircular vector field is still conformal. Some investigations are made in [2] [6]. We find that the Finslerian case is much more complicated since it is closely related to the Cartan torsion. In this paper, we will first characterize a concircular vector field by some PDEs (Theorem 6.1 below), and then using Theorem 6.1, we obtain the following Theorems 1.1 and 1.2.

Theorem 1.1 *A concircular vector field V on a Finsler manifold is conformal if and only if the Lie derivative of the mean Cartan torsion along V^c vanishes.*

Theorem 1.2 *On a Finsler manifold, a conformal vector field with the conformal factor ρ is concircular if and only if ρ satisfies*

$$\rho_{i|j} = \lambda g_{ij}, \quad \rho^r C_{ri}^k = 0, \quad (\rho_i := \rho_{x^i}, \quad \rho^i := g^{ir} \rho_r), \quad (2)$$

where $\lambda = \lambda(x)$ is a scalar function on M and the symbol $|$ means the horizontal covariant derivative of Cartan (or Chern) connection.

In (2), the horizontal covariant derivative of Cartan connection can also be replaced by that of Berwald connection due to the second equation of (2). Theorems 1.1 and 1.2 show that a concircular vector field is closely related to the Cartan torsion or mean Cartan torsion. For a Riemann metric, the Cartan torsion and the mean Cartan torsion both vanish. Then Theorems 1.1 and 1.2 show that a vector field V on a Riemann manifold is concircular iff. V is conformal with the conformal factor ρ satisfying the first formula in (2) (see [5]). In Section 7, we will see that on certain Finsler manifolds, there are concircular (resp. conformal) but not conformal (resp. concircular) vector fields.

For concircular transformations between two Finsler metrics we have the following result.

Theorem 1.3 *Let \tilde{F} and F be two conformally related Finsler metrics on a same manifold M with $\tilde{F} = u^{-1}F$. Then we have*

(i) *\tilde{F} and F are concircular if and only if*

$$u_{i|j} = \lambda g_{ij}, \quad u^r C_{ri}^k = 0, \quad (u_i := u_{x^i}, \quad u^i := g^{ir} u_r), \quad (3)$$

where $\lambda = \lambda(x)$ is a scalar function on M and the symbol $|$ means the horizontal covariant derivative of Cartan (or Chern) connection of F .

(ii) *If F and \tilde{F} are concircular, then F and \tilde{F} keep the invariance of their features of being of scalar (resp. isotropic) flag curvature, or of constant flag curvature (in $\dim(M) \geq 3$), or an Einstein metric. In this case, we have the following formula*

$$\tilde{K} = Ku^2 + 2\lambda u - u_m u^m, \quad (4)$$

where λ is given by (3), and K (resp. \tilde{K}) denotes the flag curvature or Ricci scalar of F (resp. \tilde{F}).

Theorem 1.3 (i) is an analogue of Theorem 1.2. In Theorem 1.3 (i), if F is locally Euclidean, then the local structure of \tilde{F} can be determined by solving (3), and this case can be an example to show that a geodesic (resp. circle) may be mapped to a circle (resp. geodesic) (see Remark 6.7 following the proof of Theorem 1.3). Theorem 1.3 (ii) provides a similar result as a projective map keeps scalar flag curvature unchanged. We are not sure whether the converse of Theorem 1.3 (ii) is true in dimension $n \geq 3$, which holds however at least in Riemannian case ([4]). In Theorem 1.3 (ii), if F is locally Minkowskian, then \tilde{F} is of isotropic flag curvature.

We organize the paper as follows. In Section 2, we introduce the definition of Cartan Y -connection and its basic properties. In Section 3, we introduce the definition of geodesic circles and show some basic properties of the ODE related to geodesic circles. In Section 4, we show the notion of Lie derivative and some useful formulas related to Lie derivative are given. In Section 5, we give some basic relations among conformal vector fields, projective vector fields and concircular vector fields. In Section 6, we give the proofs of our main results, and therein, we also establish the characterization theorem (Theorem 6.1) for concircular vector fields. In Section 7, we give some examples supplementary to Theorems 1.1 and 1.2.

2 Finsler Connections and Cartan Y -connection

A spray \mathbf{G} is a global vector field defined on the tangent bundle TM ,

$$\mathbf{G} := y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where G^i are called the spray (or geodesic) coefficients. Put

$$\delta_i := \frac{\partial}{\partial x^i} - G_i^r \frac{\partial}{\partial y^r}, \quad \dot{\partial}_i := \frac{\partial}{\partial y^i}, \quad \partial_i := \frac{\partial}{\partial x^i}, \quad \delta y^i := dy^i + G_r^i dx^r, \quad (G_i^r := \dot{\partial}_i G^r).$$

Then $\{\delta_i, \dot{\partial}_i\}$ is a local frame on the manifold TM and $\{dx^i, \delta y^i\}$ is its dual. We denote by $\pi : TM \mapsto M$ the natural projection. Let \mathcal{H} and \mathcal{V} be two maps from π^*TM to TTM and they are locally given by

$$\mathcal{H}v = v^i \delta_i, \quad \mathcal{V}v = v^i \dot{\partial}_i, \quad (v = v^i(x, y) \partial_i).$$

Let D be a linear connection defined on the pull-back vector bundle π^*TM with TM as the base manifold. We can put

$$D(\frac{\partial}{\partial x^i}) = \omega_i^r \frac{\partial}{\partial x^r} = (\Gamma_{ik}^r dx^k + V_{ik}^r \delta y^k) \frac{\partial}{\partial x^r}.$$

For a spray tensor T_i^j , as an example, the h - and v -covariant derivatives (denoted by $|$ and \cdot respectively) are defined respectively by

$$T_{i|k}^j := \delta_k T_i^j + T_i^r \Gamma_{rk}^j - T_r^j \Gamma_{ik}^r, \quad T_i^j \cdot_k := \dot{\partial}_k T_i^j + T_i^r V_{rk}^j - T_r^j V_{ik}^r.$$

The hh -curvature \mathbf{R} and the hv -curvature \mathbf{P} are given by

$$\begin{aligned} \mathbf{R}(X, Y)Z &:= -D_{\mathcal{H}X} D_{\mathcal{H}Y} Z + D_{\mathcal{H}Y} D_{\mathcal{H}X} Z + D_{[\mathcal{H}X, \mathcal{H}Y]} Z, \\ \mathbf{P}(X, Y)Z &:= -D_{\mathcal{H}X} D_{\mathcal{V}Y} Z + D_{\mathcal{V}Y} D_{\mathcal{H}X} Z + D_{[\mathcal{H}X, \mathcal{V}Y]} Z, \end{aligned}$$

for $X, Y, Z \in \pi^*TM$. Under the natural local basis $\{\partial_i\}$, we have

$$\begin{aligned} R_k^r{}_{ij} &= \delta_j \Gamma_{ki}^r + \Gamma_{ki}^s \Gamma_{sj}^r - \delta_i \Gamma_{kj}^r - \Gamma_{kj}^s \Gamma_{si}^r + (\delta_j G_i^s - \delta_i G_j^s) V_{ks}^r, \\ P_k^r{}_{ij} &= \dot{\partial}_j \Gamma_{ki}^r - V_{kj|i}^r - (\Gamma_{ji}^s - G_{ij}^s) V_{ks}^r, \quad (G_{ij}^s := \dot{\partial}_j G_i^s), \\ (\mathbf{R}(\partial_i, \partial_j) \partial_k &= R_k^r{}_{ij} \partial_r, \quad \mathbf{P}(\partial_i, \partial_j) \partial_k = P_k^r{}_{ij} \partial_r). \end{aligned}$$

For a Finsler metric F , there are three well-known connections: Cartan, Berwald and Chern connections, which are defined respectively by putting

$$\begin{aligned} \Gamma_{ij}^k &:= {}^* \Gamma_{ij}^k, \quad V_{ij}^k := C_{ij}^k; \quad \Gamma_{ij}^k := G_{ij}^k, \quad V_{ij}^k := 0; \quad \Gamma_{ij}^k := {}^* \Gamma_{ij}^k, \quad V_{ij}^k := 0, \\ ({}^* \Gamma_{ij}^k &:= \frac{1}{2} g^{kl} (\delta_i g_{jl} + \delta_j g_{il} - \delta_l g_{ij}), \quad G^i := \frac{1}{4} g^{il} \{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \}). \end{aligned} \quad (5)$$

In this paper we use Cartan connection D as a tool and the symbols $|$ and $|$ denote its h - and v -covariant derivatives respectively. Using Cartan connection, we can define the so called Cartan Y -connection (see [1] [8]). Let $Y = Y^i(x) \partial / \partial x^i$ be a non-zero tangent vector field on a domain of the manifold M and $g_{ij}^*(x) := g_{ij}(x, Y(x))$ be the Y -Riemannian metric induced from the vector field Y . The Cartan Y -connection (or called Barthel connection), denoted by D^* , is a linear connection on the tangent bundle TM over the base manifold M , with the connection coefficients given by

$$\begin{aligned} \Gamma_{jk}^{*i}(x) &= {}^* \Gamma_{jk}^i(x, Y(x)) + C_{jr}^i(x, Y(x)) Y_k^r(x, Y(x)), \\ (Y_j^i(x, y) &:= Y_{|j}^i(x, y) = (\partial_j Y^i)(x) + G_j^i(x, y)). \end{aligned} \quad (6)$$

We use the symbol $/$ to denote the covariant derivative of the Cartan Y -connection. For a spray tensor $T_i(x, y)$ (as an example), let $T_i^*(x) := T_i(x, Y(x))$. Then T_i^* is considered as a tensor on M and we have

$$T_{i/j}^* = (T_{i|j} + T_{i|r} Y_j^r)|_{y=Y}, \quad (7)$$

$$T_{i|j}^* = \{T_{i|j} + T_{i \cdot r} Y_j^r\}|_{y=Y}, \quad (T_{i \cdot r} := \dot{\partial}_r T_i),$$

$$T_{i|j}^* + T_i^*|_r Y_j^r = \{T_{i|j} + T_{i|r} Y_j^r\}|_{y=Y}. \quad (8)$$

Since the Cartan connection is F -metric-compatible ($g_{ij|k} = 0$, $g_{ij|k} = 0$), the Cartan Y -connection is g^* -metric-compatible ($g_{ij/k}^* = 0$) by (7).

For a Finsler manifold (M, F) and a curve $\gamma = \gamma(t)$ on M , we always in this paper let Y be a vector field in the neighborhood of γ which is an extension of $\dot{\gamma} := d\gamma/dt$ and let D^* be the Cartan Y -connection related to the vector field Y . Let $\dot{\tilde{\gamma}} := (\dot{\gamma}, \dot{\gamma})$ be the tangent vector of the curve $\tilde{\gamma} := (\gamma, \dot{\gamma})$ on TM . Then for a spray tensor $T = T_i dx^i$ we have

$$D_{\dot{\tilde{\gamma}}}^* T^* = D_{\dot{\tilde{\gamma}}} T^* = D_{\dot{\tilde{\gamma}}} T, \quad (9)$$

which follows from (note that $\dot{\gamma}^r Y_r^k = \dot{\gamma}^k + 2G^k$ and y takes the value $\dot{\gamma}$)

$$\begin{aligned} (D_{\dot{\tilde{\gamma}}}^* T^*)_i &= \dot{\gamma}^k T_{i/k}^* \stackrel{(7)}{=} \dot{\gamma}^k (T_{i|k} + T_{i|r} Y_k^r) = \dot{\gamma}^k T_{i|k} + (\dot{\gamma}^k + 2G^k) T_{i|k} \\ &= (D_{\dot{\gamma}^k \delta_k + (\dot{\gamma}^k + 2G^k) \dot{\partial}_k} T)_i = (D_{\dot{\gamma}^k \partial_k + \dot{\gamma}^k \dot{\partial}_k} T)_i = (D_{\dot{\tilde{\gamma}}} T)_i, \end{aligned}$$

and similarly $(D_{\dot{\tilde{\gamma}}}^* T^*)_i = (D_{\dot{\tilde{\gamma}}} T^*)_i$ from (8). Let U and V be two vector fields along the curve γ , and then we have (since D^* is g^* -metric-compatible)

$$\frac{d}{dt} g_{\dot{\gamma}}(U, V) = D_{\dot{\gamma}}^* g_{\dot{\gamma}}(U, V) = g_{\dot{\gamma}}(D_{\dot{\gamma}}^* U, V) + g_{\dot{\gamma}}(U, D_{\dot{\gamma}}^* V). \quad (10)$$

Remark 2.1 In (10), we can replace $D_{\dot{\gamma}}^*$ by $D_{\dot{\tilde{\gamma}}}$ from (9), but can not by $D_{\dot{\gamma}}$.

3 Geodesic circles

By a simple observation, we have the following lemma.

Proposition 3.1 *Let $\gamma = \gamma(s)$ be parameterized by the arc-length s satisfying the following ODE*

$$D_{\dot{\gamma}}^* D_{\dot{\gamma}}^* \dot{\gamma} + \tau(s) \dot{\gamma} = 0, \quad (\dot{\gamma} := d\gamma/ds),$$

where τ is a smooth function along γ . Then we have $\tau = g_{\dot{\gamma}}(D_{\dot{\gamma}}^* \dot{\gamma}, D_{\dot{\gamma}}^* \dot{\gamma})$.

Proof : By (10), we have

$$\tau = -g_{\dot{\gamma}}(D_{\dot{\gamma}}^* D_{\dot{\gamma}}^* \dot{\gamma}, \dot{\gamma}) = g_{\dot{\gamma}}(D_{\dot{\gamma}}^* \dot{\gamma}, D_{\dot{\gamma}}^* \dot{\gamma}),$$

where we have used $g_{\dot{\gamma}}(D_{\dot{\gamma}}^* \dot{\gamma}, \dot{\gamma}) = 0$ following from $g_{\dot{\gamma}}(\dot{\gamma}, \dot{\gamma}) = 1$.

Q.E.D.

Now consider a curve $\gamma = \gamma(t)$ on a Finsler manifold (M, F) satisfying the ODE

$$D_{\dot{\gamma}}^* D_{\dot{\gamma}}^* \dot{\gamma} + g_{\dot{\gamma}}(D_{\dot{\gamma}}^* \dot{\gamma}, D_{\dot{\gamma}}^* \dot{\gamma}) \dot{\gamma} = 0, \quad (\dot{\gamma} := d\gamma/dt). \quad (11)$$

For the local expansion of the first term in (11), we have

$$\begin{aligned} D_{\dot{\gamma}}^* D_{\dot{\gamma}}^* \dot{\gamma} &= D_{\dot{\gamma}}^* \left[(\ddot{\gamma}^i + 2G^i) \frac{\partial}{\partial x^i} \right] \\ &\stackrel{(6)}{=} \left[\ddot{\gamma}^i + 2(\partial_j G^i) \dot{\gamma}^j + 2G_j^i \ddot{\gamma}^j \right] \frac{\partial}{\partial x^i} + (\ddot{\gamma}^i + 2G^i) \dot{\gamma}^j (*\Gamma_{ij}^k + C_{ir}^k Y_j^r) \frac{\partial}{\partial x^k} \\ &= \left\{ \ddot{\gamma}^k + 2(\partial_j G^k) \dot{\gamma}^j - 4G_j^k G^j + (\ddot{\gamma}^i + 2G^i) [3G_i^k + C_{ir}^k (\ddot{\gamma}^r + 2G^r)] \right\} \frac{\partial}{\partial x^k}. \end{aligned} \quad (12)$$

To prove Theorems 1.1–1.3 and Theorem 6.1, we need the following Proposition 3.2.

Proposition 3.2 *Arbitrarily fix two vectors $u, v \in T_x M$ with $F(u) = 1$ and $g_u(u, v) = 0$. There is a unique curve $\gamma = \gamma(t)$ satisfying the ODE (11) with the initial condition $\gamma(0) = x, \dot{\gamma}(0) = u$ and $D_u^* \dot{\gamma} = v$. For the unique curve $\gamma = \gamma(t)$, we have $F(\dot{\gamma}) = 1$, that is, t is the arc-length parameter.*

Proof : The ODE (11) is of degree three by (12), and so the uniqueness is obvious. Put

$$f(t) := g_{\dot{\gamma}}(\dot{\gamma}, \dot{\gamma}).$$

Then by (11), it easily follows from (10) that

$$f''(t) = g(t)[1 - f(t)], \quad f(0) = 1, \quad f'(0) = 0, \quad (13)$$

where $g(t) := 2g_{\dot{\gamma}}(D_{\dot{\gamma}}^* \dot{\gamma}, D_{\dot{\gamma}}^* \dot{\gamma})$. Then by an ODE theory, (13) has a unique solution $f(t) = 1$, which implies that $F(\dot{\gamma}) = 1$. Q.E.D.

Proposition 3.3 *For the ODE (11), if $D_{\dot{\gamma}}^* \dot{\gamma} = 0$, then γ is a geodesic; if $D_{\dot{\gamma}}^* \dot{\gamma} \neq 0$, then $F(\dot{\gamma}) = 1$ iff. $g_{\dot{\gamma}}(D_{\dot{\gamma}}^* \dot{\gamma}, D_{\dot{\gamma}}^* \dot{\gamma}) = k^2$ with k being a positive constant.*

Proof : We only consider the case $D_{\dot{\gamma}}^* \dot{\gamma} \neq 0$. If $F(\dot{\gamma}) = 1$, then we have $g_{\dot{\gamma}}(D_{\dot{\gamma}}^* \dot{\gamma}, \dot{\gamma}) = 0$. Then by (10) and (11), we obtain

$$D_{\dot{\gamma}}^* g_{\dot{\gamma}}(D_{\dot{\gamma}}^* \dot{\gamma}, D_{\dot{\gamma}}^* \dot{\gamma}) = 2g_{\dot{\gamma}}(D_{\dot{\gamma}}^* D_{\dot{\gamma}}^* \dot{\gamma}, D_{\dot{\gamma}}^* \dot{\gamma}) = -2g_{\dot{\gamma}}(D_{\dot{\gamma}}^* \dot{\gamma}, D_{\dot{\gamma}}^* \dot{\gamma}) \cdot g_{\dot{\gamma}}(\dot{\gamma}, D_{\dot{\gamma}}^* \dot{\gamma}) = 0,$$

which implies $g_{\dot{\gamma}}(D_{\dot{\gamma}}^*\dot{\gamma}, D_{\dot{\gamma}}^*\dot{\gamma}) = \text{constant}$.

Conversely, if $g_{\dot{\gamma}}(D_{\dot{\gamma}}^*\dot{\gamma}, D_{\dot{\gamma}}^*\dot{\gamma}) = k^2$ is a non-zero constant, then we have

$$0 = D_{\dot{\gamma}}^*g_{\dot{\gamma}}(D_{\dot{\gamma}}^*\dot{\gamma}, D_{\dot{\gamma}}^*\dot{\gamma}) = 2g_{\dot{\gamma}}(D_{\dot{\gamma}}^*D_{\dot{\gamma}}^*\dot{\gamma}, D_{\dot{\gamma}}^*\dot{\gamma}) = -2k^2g_{\dot{\gamma}}(\dot{\gamma}, D_{\dot{\gamma}}^*\dot{\gamma}),$$

which shows $g_{\dot{\gamma}}(\dot{\gamma}, D_{\dot{\gamma}}^*\dot{\gamma}) = 0$. By this fact, further we have

$$k^2 = g_{\dot{\gamma}}(D_{\dot{\gamma}}^*\dot{\gamma}, D_{\dot{\gamma}}^*\dot{\gamma}) = D_{\dot{\gamma}}^*g_{\dot{\gamma}}(\dot{\gamma}, D_{\dot{\gamma}}^*\dot{\gamma}) - g_{\dot{\gamma}}(\dot{\gamma}, D_{\dot{\gamma}}^*D_{\dot{\gamma}}^*\dot{\gamma}) = k^2g_{\dot{\gamma}}(\dot{\gamma}, \dot{\gamma}),$$

from which we see $g_{\dot{\gamma}}(\dot{\gamma}, \dot{\gamma}) = 1$.

Q.E.D.

Remark 3.4 By Proposition 3.3, the circles of a Finsler manifold are determined by the following ODE with an initial condition:

$$D_{\dot{\gamma}}^*D_{\dot{\gamma}}^*\dot{\gamma} + k^2\dot{\gamma} = 0, \quad (F(\dot{\gamma}) = 1, \text{ or } g_{\dot{\gamma}}(D_{\dot{\gamma}}^*\dot{\gamma}, D_{\dot{\gamma}}^*\dot{\gamma}) = k^2), \quad (14)$$

where $k > 0$ is a constant. The number $1/k$ is called the radius of the circle.

In a Minkowski Finsler space, the circle equation (14) is reduced to a relatively simple form which is closely related to the Cartan torsion.

Example 3.5 Let (R^n, F) be a Minkowski space. Then by the spray $G^i = 0$ and (12), the circle equation (14) becomes

$$\ddot{\gamma}^k + \ddot{\gamma}^i \ddot{\gamma}^r C_{ir}^k(\dot{\gamma}) + k^2 \dot{\gamma}^k = 0, \quad F(\dot{\gamma}) = 1. \quad (15)$$

Example 3.6 Let (R^n, F) be a Euclidean space with $F = |y|$. Then the Cartan torsion vanishes and the circle equation (15) becomes

$$\frac{d^3 \gamma^i}{ds^3} + k^2 \frac{d\gamma^i}{ds} = 0.$$

Solving the above ODE we obtain

$$\gamma^i = a^i \cos ks + b^i \sin ks + c^i, \quad (16)$$

where a, b, c are constant vectors. It is easy to see that $F(\dot{\gamma}) = 1$ is equivalent to

$$|a| = |b| = \frac{1}{k}, \quad \langle a, b \rangle = 0. \quad (17)$$

So by (17), the curve given by (16) is a Euclidean circle in the plane spanned by the vectors a, b , with the center at the point c and the radius $1/k$.

Let $\gamma = \gamma(s)$ be a geodesic circle with s being the arc-length parameter. Then γ satisfies the ODE (1). Now let γ be parameterized by a general parameter t , and define $\gamma' := d\gamma/dt$. A simple computations shows

$$\begin{aligned} \gamma' &= F(\gamma')\dot{\gamma}, \quad D_{\gamma'}^*\gamma' = F^2(\gamma')D_{\dot{\gamma}}^*\dot{\gamma} + \frac{g_{\gamma'}(\gamma', D_{\gamma'}^*\gamma')}{F(\gamma')} \dot{\gamma}, \\ D_{\gamma'}^*D_{\gamma'}^*\gamma' &= F^3(\gamma')D_{\dot{\gamma}}^*D_{\dot{\gamma}}^*\dot{\gamma} + 3g_{\gamma'}(\gamma', D_{\gamma'}^*\gamma')D_{\dot{\gamma}}^*\dot{\gamma} + \frac{d}{dt} \left(\frac{g_{\gamma'}(\gamma', D_{\gamma'}^*\gamma')}{F(\gamma')} \right) \dot{\gamma}. \end{aligned}$$

Following the above and Proposition 3.1, we immediately obtain the following Proposition 3.7, which will be used to prove Theorem 1.3.

Proposition 3.7 *A curve $\gamma = \gamma(t)$ under a general parameter t is a geodesic circle iff. the following vector $U = U(t)$ along the curve γ ,*

$$U := D_{\gamma'}^* D_{\gamma'}^* \gamma' - 3 \frac{g_{\gamma'}(\gamma', D_{\gamma'}^* \gamma')}{F^2(\gamma')} D_{\gamma'}^* \gamma'$$

is tangent to the curve γ .

4 Lie derivatives

Consider a geometric object T on M (T is not necessarily a tensor), which is defined along curves on M with the following form

$$T = T(c) = (T_{j_1 \dots}^{i_1 \dots}(c, \dot{c}, \ddot{c}, \dots, c^{(m)})), \quad (c^{(k)} := d^k c / dt^k), \quad (18)$$

where $c = c(t)$ is an arbitrary curve parameterized by a general parameter t . Obviously, the value of T at a point $x \in M$ is dependent on the derivatives of some degrees for a curve passing through x . If $m = 0$, then T is defined along points of M , which is the case for a tensor T . The components $T_{j_1 \dots}^{i_1 \dots}$ are determined by local coordinates. For a map $f : M \mapsto M$, denote by $f_{\#}T$ the local expression of T under the local coordinate $\tilde{x} (= f(x))$ in $\tilde{U} (= f(U))$. If T is a tensor on M , then $f_{\#}$ coincides with the common map induced from the tangent map.

For a vector field V on M , it induces a flow φ_t acting on M . The Lie derivative of T along V is defined by (cf. [13] [17])

$$\mathcal{L}_V(T(c)) := \frac{d}{d\epsilon} \Big|_{\epsilon=0} [T(\varphi_{\epsilon}(c)) - \varphi_{\epsilon\#}(T(c))]. \quad (19)$$

The Lie derivative $\mathcal{L}_V T$ measures the change of T along the vector field V .

If $m = 1$ in (18), then T is actually defined along points on TM by putting $(c, \dot{c}) = (x, y)$ due to the arbitrariness of the curve $c = c(t)$, and the vector field V on M is lifted to the vector field V^c on TM , where V and V^c are locally related by

$$V = V^i \partial_i, \quad V^c = V^i \partial_i + y^r (\partial_r V^i) \dot{\partial}_i.$$

Denote by φ_t^c the flow of V^c acting on TM . Then we have a similar definition for $\mathcal{L}_{V^c} T$ as that in (19). So in this case we identify $\mathcal{L}_V T$ with $\mathcal{L}_{V^c} T$. If T is spray tensor on M , for example, $T = (T_j^i(x, y))$, by the definition (19), we easily obtain

$$\mathcal{L}_{V^c} T_j^i = V^c(T_j^i) - T_j^r (\partial_r V^i) + T_r^i (\partial_j V^r) = V^r T_{j|r}^i + V_{|0}^r T_{j \cdot r}^i - T_j^r V_{|r}^i + T_r^i V_{|j}^r, \quad (20)$$

where we have used the contraction $V_{|0}^r := V_{|i}^r y^i$.

Lemma 4.1 *For y^i , g_{ij} and G^i , we have*

$$\begin{aligned} \mathcal{L}_{V^c} y^i &= 0, & \mathcal{L}_{V^c} g_{ij} &= V_{i|j} + V_{j|i} + 2V_{|0}^r C_{rij}, & \mathcal{L}_{V^c} (2G^i) &= V_{|0|0}^i + V^r R_{\cdot r}^i, \\ (R^i_k &:= 2\partial_k G^i - y^j \partial_j G_k^i + 2G^j G_{jk}^i - G_j^i G_k^j \text{ (the Riemann curvature)}). \end{aligned}$$

Proof : We compute the last formula as follows:

$$\begin{aligned} (\mathcal{L}_{V^c} (2G^i))(x, y) &= \frac{d}{d\epsilon} \Big|_{\epsilon=0} [2G^i(\tilde{x}, \tilde{y}) - \varphi_{\epsilon\#}^c(2G^i(x, y))], \quad (\tilde{x} = \varphi_{\epsilon}(x), \tilde{y} = \varphi_{\epsilon*}(y)) \\ &= \frac{d}{d\epsilon} \Big|_{\epsilon=0} \left\{ 2G^i(\tilde{x}, \tilde{y}) - 2G^k(x, y) \frac{\partial \tilde{x}^i}{\partial x^k} + y^r y^j \frac{\partial^2 \tilde{x}^i}{\partial x^r \partial x^j} \right\} \\ &= V^c(2G^i) - 2G^r \frac{\partial V^i}{\partial x^r} + y^r y^j \frac{\partial^2 V^i}{\partial x^r \partial x^j} = V_{|0|0}^i + V^r R_{\cdot r}^i. \end{aligned}$$

The second formula follows from (20) directly.

Q.E.D.

Lemma 4.2 *For Cartan (or Chern) and Berwald connections, we have*

$$\begin{aligned} A_{jk}^i &:= \mathcal{L}_{V^c}(*\Gamma_{jk}^i) = V_{|j|k}^i + V_{|0}^r F_{j\ kr}^i + V^r K_{j\ kr}^i, \\ B_{jk}^i &:= \mathcal{L}_{V^c}(G_{jk}^i) = V_{;j;k}^i + V_{;0}^r G_{j\ kr}^i + V^r H_{j\ kr}^i, \end{aligned}$$

where $K_{j\ kr}^i$ and $F_{j\ kr}^i$ are the hh- and hv-curvatures of Chern connection, $H_{j\ kr}^i$ and $G_{j\ kr}^i$ are the hh- and hv-curvatures of Berwald connection (see Section 2), and the symbol $;$ is the h-covariant derivative of Berwald connection.

Lemma 4.3 *Related to A_{jk}^i and B_{jk}^i in Lemma 4.2, we have*

$$\begin{aligned} (\mathcal{L}_{V^c} T_i)_{|j} - \mathcal{L}_{V^c}(T_{i|j}) &= T_r A_{ij}^r + T_{i\cdot r} A_{0j}^r, & \mathcal{L}_{V^c} K_{i\ jk}^m &= A_{ij|k}^m + A_{0k}^r F_{i\ jr}^m - (j/k), \\ (\mathcal{L}_{V^c} T_i)_{;j} - \mathcal{L}_{V^c}(T_{i;j}) &= T_r B_{ij}^r + T_{i\cdot r} B_{0j}^r, & \mathcal{L}_{V^c} H_{i\ jk}^m &= B_{ij;k}^m + B_{0k}^r G_{i\ jr}^m - (j/k), \end{aligned}$$

where $T = (T_i)$ is a spray tensor (as an example), and $T_{ij} - (i/j)$ means $T_{ij} - T_{ji}$.

It is a little lengthy to prove Lemmas 4.2 and 4.3 (cf. [6]). We omit the details here.

Remark 4.4 *Acting on a general geometric object $T = (T_i(x, y))$ (as an example), $\mathcal{L}_{V^c} \partial_j = \partial_j \mathcal{L}_{V^c}$ or $\mathcal{L}_{V^c} \dot{\partial}_j = \dot{\partial}_j \mathcal{L}_{V^c}$ iff. it holds respectively ($\tilde{x} = \varphi_\epsilon(x)$)*

$$\partial_m \left\{ [\varphi_{\epsilon\#}(T_i) - T_r \frac{\partial x^r}{\partial \tilde{x}^i}]|_{\epsilon=0} \right\} \cdot \frac{\partial V^m}{\partial x^j} = 0, \quad \dot{\partial}_m \left\{ [\varphi_{\epsilon\#}(T_i) - T_r \frac{\partial x^r}{\partial \tilde{x}^i}]|_{\epsilon=0} \right\} \cdot \frac{\partial V^m}{\partial x^j} = 0.$$

If T is a spray tensor or T is the spray G^i , the above conditions are satisfied, because for a spray tensor T_i and the spray G^i we respectively have

$$\varphi_{\epsilon\#}(T_i) - T_r \frac{\partial x^r}{\partial \tilde{x}^i} = 0, \quad [\varphi_{\epsilon\#}(G^i) - G^r \frac{\partial \tilde{x}^i}{\partial x^r}]_{\epsilon=0} = \left[-\frac{1}{2} \frac{\partial^2 \tilde{x}^i}{\partial x^r \partial x^m} y^r y^m \right]_{\epsilon=0} = 0.$$

For a curve $c = c(t)$ with a general parameter t , by the definition (19), we have

$$\mathcal{L}_V c' = \frac{d}{d\epsilon}(\tilde{c}' - \varphi_{\epsilon\#} c') = \frac{d}{d\epsilon}(\tilde{c}' - \tilde{c}') = 0, \quad \mathcal{L}_V c'' = 0, \quad \dots \quad (21)$$

Now consider a geometric object T on M defined along curves, in the following form

$$T = T(c) = (T_{j_1 \dots}^{i_1 \dots}(c, \dot{c}, \ddot{c}, \dots, c^{(m)})), \quad (c^{(k)} := d^k c / ds^k), \quad (22)$$

where s is the arc-length parameter. When we consider the covariant derivative of T or $\mathcal{L}_V T$ defined along curves, the understanding is to regard T or $\mathcal{L}_V T$ as a new object $T^* = T^*(x)$ defined along points in a neighborhood of the curve c . That is, let Y be a vector field in the neighborhood of c which is an extension of $\dot{c} := dc/ds$, and then taking $c(s) = x$, $\dot{c}^i(s) = Y^i(x)$, $\ddot{c}^i(s) = Y^r \partial_r Y^i, \dots$, we obtain a new geometric object $T^* = T^*(x)$ from T in a neighborhood of the curve c . But we should keep in mind that the Lie derivative always acts on an object defined along curves. We will use the following Proposition 4.5 to prove Theorems 6.1 below.

Proposition 4.5 *Let V be a vector field on M . For a geometric object $T = (T_i)$ on M defined by (22), we have the following exchanging formulas,*

$$\dot{c}^k \mathcal{L}_V(T_{j/k}) = \dot{c}^k (\mathcal{L}_V T_j)_{/k} - \dot{c}^k T_r A_{jk}^{*r}, \quad (A_{jk}^{*r} := \mathcal{L}_V(\Gamma_{jk}^{*r})), \quad (23)$$

$$\dot{c}^k \mathcal{L}_V(T_{j|k}) = \dot{c}^k (\mathcal{L}_V T_j)_{|k} - \dot{c}^k T_r A_{jk}^r. \quad (24)$$

Proof : Note that Lemma 4.3 is of no help in this proof, and the Lie derivative and all values are taken along the curve c . We only prove (23) for the Cartan Y -connection (it is similar for (24)). Let t be a general parameter of c with $c' := dc/dt$, and we have

$$\mathcal{L}_V \dot{c}^k = \mathcal{L}_V(F^{-1}(c'^k) c'^k) = (\mathcal{L}_{V^c} F^{-1}) c'^k = -(\mathcal{L}_{V^c} \ln F) \dot{c}^k, \quad (25)$$

where we have used $\mathcal{L}_V c'^k = 0$ by (21); or in another way we have

$$\mathcal{L}_V \dot{c}^k = \mathcal{L}_{V^c}(F^{-1}(y)y^k) = (\mathcal{L}_{V^c} F^{-1})y^k = -(\mathcal{L}_{V^c} \ln F) \dot{c}^k.$$

Now by $T_{j/k} = \partial_k T_j - T_r \Gamma_{jk}^{*r}$ we have

$$\dot{c}^k \mathcal{L}_V(T_{j/k}) = \dot{c}^k \mathcal{L}_V(\partial_k T_j) - \dot{c}^k \mathcal{L}_V(T_r \Gamma_{jk}^{*r}). \quad (26)$$

In the right hand side of (26), the first term is written as

$$\begin{aligned} \dot{c}^k \mathcal{L}_V(\partial_k T_j) &= \mathcal{L}_V(\dot{c}^k \partial_k T_j) - (\partial_k T_j)(\mathcal{L}_V \dot{c}^k) \stackrel{(25)}{=} \mathcal{L}_V\left(\frac{d}{ds} T_j\right) + (\mathcal{L}_{V^c} \ln F) \frac{d}{ds} T_j \\ &= \frac{d}{ds} (\mathcal{L}_V T_j), \end{aligned}$$

the last equality of which follows from

$$\begin{aligned} \mathcal{L}_V\left(\frac{d}{ds} T_j\right) &= \mathcal{L}_V(F^{-1} \frac{d}{dt} T_j) = (\mathcal{L}_{V^c} F^{-1}) \frac{d}{dt} T_j + F^{-1} (\mathcal{L}_V \frac{d}{dt} T_j) \\ &= -(\mathcal{L}_{V^c} \ln F) \frac{d}{ds} T_j + F^{-1} \frac{d}{dt} \mathcal{L}_V T_j \\ &= -(\mathcal{L}_{V^c} \ln F) \frac{d}{ds} T_j + \frac{d}{ds} \mathcal{L}_V T_j. \end{aligned}$$

Thus (26) gives

$$\dot{c}^k \mathcal{L}_V(T_{j/k}) = \frac{d}{ds} (\mathcal{L}_V T_j) - \dot{c}^k \mathcal{L}_V(T_r \Gamma_{jk}^{*r}). \quad (27)$$

On the other hand we have

$$\dot{c}^k (\mathcal{L}_V T_j)_{/k} = \dot{c}^k \partial_k (\mathcal{L}_V T_j) - (\mathcal{L}_V T_r) \dot{c}^k \Gamma_{jk}^{*r} = \frac{d}{ds} (\mathcal{L}_V T_j) - (\mathcal{L}_V T_r) \dot{c}^k \Gamma_{jk}^{*r} \quad (28)$$

Then (27) – (28) gives

$$\begin{aligned} -\dot{c}^k \mathcal{L}_V(T_r \Gamma_{jk}^{*r}) + (\mathcal{L}_V T_r) \dot{c}^k \Gamma_{jk}^{*r} &= -\dot{c}^k [(\mathcal{L}_V T_r) \Gamma_{jk}^{*r} + T_r (\mathcal{L}_V \Gamma_{jk}^{*r})] + (\mathcal{L}_V T_r) \dot{c}^k \Gamma_{jk}^{*r} \\ &= -\dot{c}^k T_r (\mathcal{L}_V \Gamma_{jk}^{*r}). \end{aligned}$$

This gives the proof of (23).

Q.E.D.

5 Some vector fields

Using Lie derivative, we can characterize some special vector fields.

(A). Conformal vector fields. A vector field V is conformal iff. it satisfies

$$\mathcal{L}_{V^c} g_{ij} = 2\rho g_{ij}, \quad (\rho = \rho(x) \text{ on } M). \quad (29)$$

The scalar function ρ is just the conformal factor. V is homothetic iff. ρ is a constant. V is Killing iff. $\rho = 0$.

(B). Projective vector fields. A vector field V is projective iff. it satisfies

$$\mathcal{L}_{V^c}(2G^i) = Py^i, \quad (P = P(x, y) \text{ on } TM). \quad (30)$$

The function P is the projective factor. V is affine iff. $P = 0$. Geodesics keep invariant under the flow of a projective vector field. A projective vector field V can also be characterized by the following equation

$$\mathcal{L}_V U = 0, \quad (U := D_{\dot{\gamma}}^* \dot{\gamma} = 0, \quad \dot{\gamma} = d\gamma/ds),$$

where $\gamma = \gamma(s)$ is an arbitrary curve with s being the arc-length parameter. A direct computation shows that under the condition $U = 0$, we have (see (38) below)

$$\mathcal{L}_V U^k = -\frac{d}{ds}(\mathcal{L}_{V^c} \ln F) \dot{\gamma}^k + A_{mr}^k \dot{\gamma}^m \dot{\gamma}^r.$$

Thus $\mathcal{L}_V U^k = 0$ for any $\dot{\gamma}^k$ gives $A_{00}^k = Py^k$, which is just (30) by Lemmas 4.1 and 4.2.

(C). Concircular vector fields. Unlike the characterization on TM for conformal and projective vector fields (see (29) and (30)), we use Lie derivative acting on a geometric object defined along curves to characterize a concircular vector field. A concircular vector field V is characterized by the following equation

$$\mathcal{L}_V U = 0, \quad (U := D_{\dot{\gamma}}^* D_{\dot{\gamma}}^* \dot{\gamma} + g_{\dot{\gamma}}(D_{\dot{\gamma}}^* \dot{\gamma}, D_{\dot{\gamma}}^* \dot{\gamma}) \dot{\gamma} = 0, \quad \dot{\gamma} = d\gamma/ds), \quad (31)$$

where $\gamma = \gamma(s)$ is an arbitrary curve with s being the arc-length parameter. We will compute $\mathcal{L}_V U$ in the next section in the proof of Theorem 6.1 below.

By definition, a projective vector field is not necessarily concircular, and a concircular vector field is not necessarily projective, either.

It is verified in [14] that a conformal and projective vector field is homothetic, and a homothetic vector field is affine.

For a Riemannian manifold (M, F) , a concircular vector field is conformal ([5]). It is shown in [16] that if (M, F) is an Einstein-Riemann manifold with the dimension $n \geq 3$, the conformal factor ρ of a conformal vector field satisfies $\rho_{i|j} = \lambda g_{ij}$ for a scalar function $\lambda = \lambda(x)$. This fact can follow from Lemma 4.3. Then we conclude from a result in [5] or Theorems 1.1 and 1.2 that, *on an Einstein-Riemann manifold of dimension $n \geq 3$, conformal vector fields coincide with concircular vector fields*. By this and a result in [18], we see that, *on a Riemann manifold of constant sectional curvature in $n \geq 3$, the local structure of concircular vector fields can be determined completely*.

6 Proofs of main results

6.1 Characterization of concircular vector fields

Before we prove Theorems 1.1 and 1.2, we first give a characterization for concircular vector fields by some PDEs. Define some spray tensors on TM as follows:

$$T_{ij}^k := 3[(\mathcal{L}_{V^c} y_i) \delta_j^k + (i/j)] - 2F^2 \mathcal{L}_{V^c} C_{ij}^k - 2(\mathcal{L}_{V^c} y_r) C_{ij}^r y^k, \quad (32)$$

$$\theta_i^k := 3[F^2(\mathcal{L}_{V^c} y_i) - V^c(F^2) y_i] y^k + 3F^2 V^c(F^2) \delta_i^k, \quad (33)$$

$$S^k := -[V^c(F^2)]_{|0|0} y^k + 2F^2 A_{00|0}^k, \quad (34)$$

$$\begin{aligned} Z_i^k &:= \{4A_{00}^r g_{ir} - [V^c(F^2)]_{|i} - 4(\mathcal{L}_{V^c} y_i)_{|0}\} y^k - 3[V^c(F^2)]_{|0} \delta_i^k \\ &\quad + 2F^2(3A_{i0}^k + 2A_{00}^r C_{ir}^k), \end{aligned} \quad (35)$$

$$\lambda^k := 4[A_{00}^r y_r - 2[V^c(F^2)]_{|0}] y^k + 6F^2 A_{00}^k. \quad (36)$$

Theorem 6.1 *Let V be a vector field on a Finsler manifold (M, F) . Then V is concircular iff. V satisfies the following PDEs on the tangent bundle TM :*

$$F^4 T_{ij}^k = y_i \theta_j^k + y_j \theta_i^k, \quad S^k = 0, \quad F^2 Z_i^k = \lambda^k y_i, \quad (37)$$

where $T_{ij}^k, \theta_i^k, S^k, Z_i^k$ and λ^k are given by (32)–(36).

Let $\gamma = \gamma(s)$ be an arbitrary curve with s being the arc-length parameter. We will use the Cartan Y -connection as a tool, where Y is a vector field as an extension of $\dot{\gamma}$ in a neighborhood of γ . Since a concircular vector field V is characterized by (31), to prove Theorem 6.1, we need to first compute $\mathcal{L}_V U$. Note that in the following Lemmas 6.2–6.4, all quantities take values along the curve γ . For example, we have $\mathcal{L}_V C_{ij}^k = \mathcal{L}_V [C_{ij}^k(\gamma(s), \dot{\gamma}(s))] \neq \mathcal{L}_{V^c} C_{ij}^k$, but we have $\mathcal{L}_V g_{ij} = \mathcal{L}_{V^c} g_{ij}$ due to the zero-homogeneity of g_{ij} .

Lemma 6.2 *For two terms in $\mathcal{L}_V U$ we have*

$$\begin{aligned} \mathcal{L}_V (D_{\dot{\gamma}}^* \dot{\gamma})^k &= -2(\mathcal{L}_{V^c} \ln F)(D_{\dot{\gamma}}^* \dot{\gamma})^k - \frac{d}{ds}(\mathcal{L}_{V^c} \ln F) \dot{\gamma}^k + A_{rm}^k \dot{\gamma}^r \dot{\gamma}^m, \\ \mathcal{L}_V (D_{\dot{\gamma}}^* D_{\dot{\gamma}}^* \dot{\gamma})^k &= -3(\mathcal{L}_{V^c} \ln F)(D_{\dot{\gamma}}^* D_{\dot{\gamma}}^* \dot{\gamma})^k - 3 \frac{d}{ds}(\mathcal{L}_{V^c} \ln F)(D_{\dot{\gamma}}^* \dot{\gamma})^k - \frac{d^2}{ds^2}(\mathcal{L}_{V^c} \ln F) \dot{\gamma}^k \\ &\quad + 3A_{rm}^k (D_{\dot{\gamma}}^* \dot{\gamma})^r \dot{\gamma}^m + A_{rm|i}^k \dot{\gamma}^r \dot{\gamma}^m \dot{\gamma}^i + 2A_{rm}^p C_{pi}^k (D_{\dot{\gamma}}^* \dot{\gamma})^i \dot{\gamma}^r \dot{\gamma}^m \\ &\quad - (\mathcal{L}_{V^c} \ln F) C_{rm}^k (D_{\dot{\gamma}}^* \dot{\gamma})^r (D_{\dot{\gamma}}^* \dot{\gamma})^m + (\mathcal{L}_V C_{rm}^k) (D_{\dot{\gamma}}^* \dot{\gamma})^r (D_{\dot{\gamma}}^* \dot{\gamma})^m. \end{aligned} \quad (38)$$

Proof : First note that

$$\dot{\gamma}^r Y_r^k = (D_{\dot{\gamma}}^* \dot{\gamma})^k, \quad \mathcal{L}_V \dot{\gamma}^k = -(\mathcal{L}_{V^c} \ln F) \dot{\gamma}^k, \quad \dot{\gamma}^m \mathcal{L}_V (C_{mr}^k Y_i^r) = 0 \text{ (by } \dot{\gamma}^m C_{mr}^k = 0).$$

By (23) in Proposition 4.5, we have

$$\begin{aligned} \mathcal{L}_V (D_{\dot{\gamma}}^* \dot{\gamma})^k &= \mathcal{L}_V (\dot{\gamma}^i \dot{\gamma}_{/i}^k) = (\mathcal{L}_V \dot{\gamma}^i) \dot{\gamma}_{/i}^k + \dot{\gamma}^i \mathcal{L}_V (\dot{\gamma}_{/i}^k) \\ &= -(\mathcal{L}_{V^c} \ln F) \dot{\gamma}^i \dot{\gamma}_{/i}^k + \dot{\gamma}^i (\mathcal{L}_V \dot{\gamma}_{/i}^k) + \dot{\gamma}^i \dot{\gamma}^m [A_{mi}^k + \mathcal{L}_V (C_{mr}^k Y_i^r)], \end{aligned}$$

which immediately gives (38). To prove (39), first we have

$$\begin{aligned} \mathcal{L}_V (D_{\dot{\gamma}}^* D_{\dot{\gamma}}^* \dot{\gamma})^k &= \mathcal{L}_V [\dot{\gamma}^i (D_{\dot{\gamma}}^* \dot{\gamma})_{/i}^k] = (\mathcal{L}_V \dot{\gamma}^i) (D_{\dot{\gamma}}^* \dot{\gamma})_{/i}^k + \dot{\gamma}^i \mathcal{L}_V [(D_{\dot{\gamma}}^* \dot{\gamma})_{/i}^k] \\ &= -(\mathcal{L}_{V^c} \ln F) (D_{\dot{\gamma}}^* D_{\dot{\gamma}}^* \dot{\gamma})^k + \dot{\gamma}^i \mathcal{L}_V [(D_{\dot{\gamma}}^* \dot{\gamma})_{/i}^k]. \end{aligned} \quad (40)$$

By (23) and then by (38), the second term in the right hand side of (40) is given by

$$\begin{aligned} \dot{\gamma}^i \mathcal{L}_V [(D_{\dot{\gamma}}^* \dot{\gamma})_{/i}^k] &= \dot{\gamma}^i [\mathcal{L}_V (D_{\dot{\gamma}}^* \dot{\gamma})^k]_{/i} + \dot{\gamma}^i (D_{\dot{\gamma}}^* \dot{\gamma})^m [A_{mi}^k + \mathcal{L}_V (C_{mr}^k Y_i^r)] \\ &= -2(\mathcal{L}_{V^c} \ln F) (D_{\dot{\gamma}}^* D_{\dot{\gamma}}^* \dot{\gamma})^k - 3 \frac{d}{ds}(\mathcal{L}_{V^c} \ln F) (D_{\dot{\gamma}}^* \dot{\gamma})^k - \frac{d^2}{ds^2}(\mathcal{L}_{V^c} \ln F) \dot{\gamma}^k \\ &\quad + 3A_{rm}^k (D_{\dot{\gamma}}^* \dot{\gamma})^r \dot{\gamma}^m + A_{rm|i}^k \dot{\gamma}^r \dot{\gamma}^m \dot{\gamma}^i + (D_{\dot{\gamma}}^* \dot{\gamma})^m \dot{\gamma}^i \mathcal{L}_V (C_{mr}^k Y_i^r). \end{aligned} \quad (41)$$

For the last two terms in (41) we have

$$\begin{aligned} A_{rm|i}^k \dot{\gamma}^r \dot{\gamma}^m \dot{\gamma}^i &= A_{rm|i}^k \dot{\gamma}^r \dot{\gamma}^m \dot{\gamma}^i + A_{rm}^p Y_i^p \dot{\gamma}^r \dot{\gamma}^m \dot{\gamma}^i \\ &= A_{rm|i}^k \dot{\gamma}^r \dot{\gamma}^m \dot{\gamma}^i + A_{rm}^p C_{pi}^k \dot{\gamma}^r \dot{\gamma}^m (D_{\dot{\gamma}}^* \dot{\gamma})^i, \end{aligned} \quad (42)$$

$$\dot{\gamma}^i \mathcal{L}_V (C_{mr}^k Y_i^r) = \mathcal{L}_V ((D_{\dot{\gamma}}^* \dot{\gamma})^r C_{mr}^k) + (\mathcal{L}_{V^c} \ln F) C_{mr}^k (D_{\dot{\gamma}}^* \dot{\gamma})^r. \quad (43)$$

By (38) we have

$$\begin{aligned}\mathcal{L}_V((D_\gamma^*\dot{\gamma})^r C_{mr}^k) &= (D_\gamma^*\dot{\gamma})^r \mathcal{L}_V C_{mr}^k + C_{mr}^k \mathcal{L}_V (D_\gamma^*\dot{\gamma})^r \\ &= (D_\gamma^*\dot{\gamma})^r \mathcal{L}_V C_{mr}^k + C_{mr}^k [-2(\mathcal{L}_{V^c} \ln F)(D_\gamma^*\dot{\gamma})^r + A_{ip}^r \dot{\gamma}^i \dot{\gamma}^p].\end{aligned}\quad (44)$$

Now plugging (44) into (43), then (42) and (43) into (41), and then (41) into (40), we finally obtain (39). Q.E.D.

Lemma 6.3 *The equation $\mathcal{L}_V U = 0$ in (31) is equivalent to (under the condition $U = 0$)*

$$\begin{aligned}0 &= \left\{ [\mathcal{L}_V g_{ij} - 2(\mathcal{L}_{V^c} \ln F) g_{ij}] (D_\gamma^*\dot{\gamma})^i (D_\gamma^*\dot{\gamma})^j - \frac{d^2}{ds^2} (\mathcal{L}_{V^c} \ln F) + 2g_{ij} A_{rm}^i \dot{\gamma}^r \dot{\gamma}^m (D_\gamma^*\dot{\gamma})^j \right\} \dot{\gamma}^k \\ &\quad - 3 \frac{d}{ds} (\mathcal{L}_{V^c} \ln F) (D_\gamma^*\dot{\gamma})^k + 3A_{rm}^k (D_\gamma^*\dot{\gamma})^r \dot{\gamma}^m + A_{rm|i}^k \dot{\gamma}^r \dot{\gamma}^m \dot{\gamma}^i + 2A_{rm}^p C_{pi}^k (D_\gamma^*\dot{\gamma})^i \dot{\gamma}^r \dot{\gamma}^m \\ &\quad - (\mathcal{L}_{V^c} \ln F) C_{rm}^k (D_\gamma^*\dot{\gamma})^r (D_\gamma^*\dot{\gamma})^m + (\mathcal{L}_V C_{rm}^k) (D_\gamma^*\dot{\gamma})^r (D_\gamma^*\dot{\gamma})^m.\end{aligned}\quad (45)$$

Proof : First by the definition of U we have

$$\begin{aligned}\mathcal{L}_V U^k &= \mathcal{L}_V (D_\gamma^* D_\gamma^* \dot{\gamma})^k + \{ (\mathcal{L}_V g_{ij}) (D_\gamma^* \dot{\gamma})^i (D_\gamma^* \dot{\gamma})^j + 2g_{ij} (D_\gamma^* \dot{\gamma})^j \mathcal{L}_V (D_\gamma^* \dot{\gamma})^i \\ &\quad - (\mathcal{L}_{V^c} \ln F) g_{ij} (D_\gamma^* \dot{\gamma})^i (D_\gamma^* \dot{\gamma})^j \} \dot{\gamma}^k.\end{aligned}\quad (46)$$

Plugging (38), (39) and

$$(D_\gamma^* D_\gamma^* \dot{\gamma})^k = -g_{ij} (D_\gamma^* \dot{\gamma})^i (D_\gamma^* \dot{\gamma})^j \dot{\gamma}^k$$

into (46), we immediately obtain (45) from $\mathcal{L}_V U = 0$. Q.E.D.

To simplify (45), we rewrite (45) in a different form in the following lemma.

Lemma 6.4 *Put $X := D_\gamma^* \dot{\gamma}$. Then (45) is equivalent to*

$$\tilde{T}_{ij}^k X^i X^j + \tilde{Z}_i^k X^i + \tilde{S}^k = 0, \quad (47)$$

where \tilde{T}_{ij}^k , \tilde{Z}_i^k and \tilde{S}^k are defined by

$$\begin{aligned}\tilde{T}_{ij}^k &:= -\frac{3}{2} [(\mathcal{L}_V g_{ir}) \dot{\gamma}^r \delta_j^k + (i/j)] + \mathcal{L}_V C_{ij}^k - (\mathcal{L}_{V^c} \ln F) C_{ij}^k + (\mathcal{L}_V g_{rm}) C_{ij}^r \dot{\gamma}^m \dot{\gamma}^k, \\ \tilde{Z}_i^k &:= [2A_{rm}^p g_{ip} - \frac{1}{2} (\mathcal{L}_V g_{rm})_{|i} - 2(\mathcal{L}_V g_{ir})_{|m}] \dot{\gamma}^m \dot{\gamma}^r \dot{\gamma}^k - \frac{3}{2} (\mathcal{L}_V g_{rm})_{|p} \dot{\gamma}^m \dot{\gamma}^r \dot{\gamma}^p \delta_i^k \\ &\quad 3A_{ir}^k \dot{\gamma}^r + 2A_{rm}^p C_{ip}^k \dot{\gamma}^m \dot{\gamma}^r, \\ \tilde{S}^k &:= [-\frac{1}{2} (\mathcal{L}_V g_{ij})_{|r|m} \dot{\gamma}^j \dot{\gamma}^k + A_{rm|i}^k] \dot{\gamma}^m \dot{\gamma}^r \dot{\gamma}^i.\end{aligned}$$

Proof : It needs to expand the derivatives of $\mathcal{L}_{V^c} \ln F = (\mathcal{L}_V g_{ij}) \dot{\gamma}^i \dot{\gamma}^j / 2$ with respect to s in (45). We have the following direct results:

$$\frac{d}{ds} (\mathcal{L}_{V^c} \ln F) = \frac{1}{2} (\mathcal{L}_V g_{ij})_{|r} \dot{\gamma}^i \dot{\gamma}^j \dot{\gamma}^r + (\mathcal{L}_V g_{ij}) \dot{\gamma}^i (D_\gamma^* \dot{\gamma})^j, \quad (48)$$

$$\begin{aligned}\frac{d^2}{ds^2} (\mathcal{L}_{V^c} \ln F) &= [\mathcal{L}_V g_{ij} - 2(\mathcal{L}_{V^c} \ln F) g_{ij}] (D_\gamma^* \dot{\gamma})^i (D_\gamma^* \dot{\gamma})^j + \frac{1}{2} (\mathcal{L}_V g_{ij})_{|r|m} \dot{\gamma}^i \dot{\gamma}^j \dot{\gamma}^r \dot{\gamma}^m \\ &\quad + \frac{1}{2} (\mathcal{L}_V g_{ij})_{|r} \dot{\gamma}^i \dot{\gamma}^j (D_\gamma^* \dot{\gamma})^r + 2(\mathcal{L}_V g_{ij})_{|r} \dot{\gamma}^i \dot{\gamma}^r (D_\gamma^* \dot{\gamma})^j \\ &\quad - (\mathcal{L}_V g_{ij}) C_{rm}^i \dot{\gamma}^j (D_\gamma^* \dot{\gamma})^r (D_\gamma^* \dot{\gamma})^m.\end{aligned}\quad (49)$$

Plugging (48) and (49) into (45), we immediately obtain (47). We can conclude (48) and (49) in the following way. First it is easy to see that

$$(\mathcal{L}_V g_{ij})|_m y^i y^j = 0, \quad (\mathcal{L}_V g_{ij})|_r y^i y^j y^r = (\mathcal{L}_V g_{ij})|_{m|r} y^i y^j y^r = 0. \quad (50)$$

Using Cartan Y-connection, we see

$$\frac{d}{ds}(\mathcal{L}_{V^c} \ln F) = \frac{1}{2}(\mathcal{L}_V g_{ij})_{/r} \dot{\gamma}^i \dot{\gamma}^j \dot{\gamma}^r + (\mathcal{L}_V g_{ij}) \dot{\gamma}^i (D_{\dot{\gamma}}^* \dot{\gamma})^j,$$

which gives (48) from (7) and (50). To show (49), we first have

$$\begin{aligned} & -\frac{d^2}{ds^2}(\mathcal{L}_{V^c} \ln F) + [\mathcal{L}_V g_{ij} - 2(\mathcal{L}_{V^c} \ln F)g_{ij}](D_{\dot{\gamma}}^* \dot{\gamma})^i (D_{\dot{\gamma}}^* \dot{\gamma})^j \\ &= -\frac{1}{2}(\mathcal{L}_V g_{ij})_{/r/m} \dot{\gamma}^i \dot{\gamma}^j \dot{\gamma}^r \dot{\gamma}^m - \frac{1}{2}(\mathcal{L}_V g_{ij})_{/r} \dot{\gamma}^i \dot{\gamma}^j (D_{\dot{\gamma}}^* \dot{\gamma})^r - 2(\mathcal{L}_V g_{ij})_{/r} \dot{\gamma}^i \dot{\gamma}^r (D_{\dot{\gamma}}^* \dot{\gamma})^j. \end{aligned} \quad (51)$$

For the three terms in the right hand side of (51), we rewrite them as follows. By (7) and (50), we easily get

$$(\mathcal{L}_V g_{ij})_{/r} \dot{\gamma}^i \dot{\gamma}^j = (\mathcal{L}_V g_{ij})|_r \dot{\gamma}^i \dot{\gamma}^j.$$

By (7) and

$$(\mathcal{L}_V g_{ij})|_m \dot{\gamma}^i = [2\mathcal{L}_V C_{ijm} - (\mathcal{L}_V g_{rj})C_{im}^r - (\mathcal{L}_V g_{ir})C_{jm}^r] \dot{\gamma}^i = -(\mathcal{L}_V g_{ir})C_{jm}^r \dot{\gamma}^i, \quad (52)$$

we have

$$\begin{aligned} (\mathcal{L}_V g_{ij})_{/r} \dot{\gamma}^i \dot{\gamma}^r (D_{\dot{\gamma}}^* \dot{\gamma})^j &= (\mathcal{L}_V g_{ij})|_r \dot{\gamma}^i \dot{\gamma}^r (D_{\dot{\gamma}}^* \dot{\gamma})^j + (\mathcal{L}_V g_{ij})|_m Y_r^m \dot{\gamma}^i \dot{\gamma}^r (D_{\dot{\gamma}}^* \dot{\gamma})^j \\ &= (\mathcal{L}_V g_{ij})|_r \dot{\gamma}^i \dot{\gamma}^r (D_{\dot{\gamma}}^* \dot{\gamma})^j + (\mathcal{L}_V g_{ij})|_m \dot{\gamma}^i (D_{\dot{\gamma}}^* \dot{\gamma})^m (D_{\dot{\gamma}}^* \dot{\gamma})^j \\ &= (\mathcal{L}_V g_{ij})|_r \dot{\gamma}^i \dot{\gamma}^r (D_{\dot{\gamma}}^* \dot{\gamma})^j - (\mathcal{L}_V g_{ir})C_{jm}^r \dot{\gamma}^i (D_{\dot{\gamma}}^* \dot{\gamma})^m (D_{\dot{\gamma}}^* \dot{\gamma})^j. \end{aligned}$$

Finally for the first term of (51), we have

$$\begin{aligned} (\mathcal{L}_V g_{ij})_{/r/m} \dot{\gamma}^i \dot{\gamma}^j \dot{\gamma}^r \dot{\gamma}^m &= [(\mathcal{L}_V g_{ij})|_r + (\mathcal{L}_V g_{ij})|_p Y_r^p]_{/m} \dot{\gamma}^i \dot{\gamma}^j \dot{\gamma}^r \dot{\gamma}^m \\ &= \left\{ (\mathcal{L}_V g_{ij})|_{r|m} + (\mathcal{L}_V g_{ij})|_r |_p Y_m^p + (\mathcal{L}_V g_{ij})|_p Y_{r/m}^p \right. \\ &\quad \left. + [(\mathcal{L}_V g_{ij})|_{p|m} + (\mathcal{L}_V g_{ij})|_p |_q Y_m^q] Y_r^p \right\} \dot{\gamma}^i \dot{\gamma}^j \dot{\gamma}^r \dot{\gamma}^m \\ &\stackrel{(50)}{=} [(\mathcal{L}_V g_{ij})|_{r|m} + (\mathcal{L}_V g_{ij})|_p |_q Y_m^q Y_r^p] \dot{\gamma}^i \dot{\gamma}^j \dot{\gamma}^r \dot{\gamma}^m \\ &= (\mathcal{L}_V g_{ij})|_{r|m} \dot{\gamma}^i \dot{\gamma}^j \dot{\gamma}^r \dot{\gamma}^m + 2(\mathcal{L}_V g_{ij})C_{rm}^i \dot{\gamma}^j (D_{\dot{\gamma}}^* \dot{\gamma})^r (D_{\dot{\gamma}}^* \dot{\gamma})^m, \end{aligned}$$

in which, the last equality follows from

$$\begin{aligned} (\mathcal{L}_V g_{ij})|_p |_q y^i y^j &= [(\mathcal{L}_V g_{ij})|_p y^i y^j]|_q - 2(\mathcal{L}_V g_{qj})|_p y^j \\ &\stackrel{(50)}{=} -2(\mathcal{L}_V g_{qj})|_p y^j \stackrel{(52)}{=} 2(\mathcal{L}_V g_{ij})y^j C_{pq}^i. \end{aligned}$$

Thus we obtain (49) from (51). Q.E.D.

Lemma 6.5 *In an n -dimensional inner product space with the metric matrix (g_{ij}) , let $a_{ij}v^i v^j = 0$ be a quadratic-form equation holding for arbitrary $v \in U^\perp$, where U^\perp is an $(n-1)$ -dimensional space perpendicular to a unit vector $u = (u^i)$. Then we have*

$$\begin{aligned} a_{ij} &= \theta_i u_j + \theta_j u_i, \\ (\theta_i &:= a_{ir} u^r - \frac{1}{2} a_{rm} u^r u^m u_i, \quad u_i := g_{ij} u^j). \end{aligned} \quad (53)$$

Proof : First we have

$$0 = a_{ij}(v^i + \bar{v}^i)(v^j + \bar{v}^j) = 2a_{ij}v^i\bar{v}^j, \quad (\forall v \in U^\perp, \bar{v} \in U^\perp),$$

which implies $a_{ij}v^j = \lambda u_i$ for some $\lambda = \lambda(\bar{v})$. Since $u = (u^i)$ is a unit vector, we easily get $\lambda = a_{ij}u^i v^j$. Thus $a_{ij}v^j = \lambda u_i$ is written as

$$(a_{ij} - a_{jr}u^r u_i)v^j = 0, \quad (\forall v \in U^\perp),$$

which gives

$$a_{ij} - a_{jr}u^r u_i = \tau_i u_j, \quad (\text{for some } \tau = (\tau_i)). \quad (54)$$

Contracting both sides of (54) by u^j we get

$$\tau_i = a_{ir}u^r - a_{rm}u^r u^m u_i.$$

Plugging the above τ_i into (54) we have

$$a_{ij} = a_{jr}u^r u_i + a_{ir}u^r u_j - a_{rm}u^r u^m u_i u_j = (a_{jr}u^r - \frac{1}{2}a_{rm}u^r u^m u_j)u_i + (i/j),$$

which gives (53). Q.E.D.

Proof of Theorem 6.1 : Let (M, F) be an n -dimensional Finsler manifold. By the definition of a geodesic circle and Proposition 3.2, we know that for any two vectors $u, v \in T_x M$ with $F(u) = 1$ and $g_u(u, v) = 0$, there is a unique geodesic circle $\gamma = \gamma(s)$ satisfying $F(\dot{\gamma}(s)) = 1, \gamma(0) = u$ and $D_{\dot{\gamma}(0)}^* \dot{\gamma} = v$. Now a vector field V is concircular iff. V satisfies (31) for any curve $\gamma = \gamma(s)$ with s being the arc-length.

We only need to prove (37) under the assumption that V is a concircular vector field. Then at an arbitrarily fixed point $x \in M$ and unit vector $u := \dot{\gamma}(0) \in T_x M$, by Lemma 6.4 together with Proposition 3.2, we have (47) for arbitrary $X \in U^\perp$, where U^\perp is an $(n-1)$ -dimensional space perpendicular to u under the inner product g_u . So (47) is considered as a polynomial of $X \in U^\perp$ and it is equivalent to

$$\tilde{T}_{ij}^k X^i X^j = 0, \quad \tilde{Z}_i^k X^i = 0, \quad \tilde{S}^k = 0. \quad (55)$$

Note that X does not belong to the total space $T_x M$ and so generally we don't have $\tilde{T}_{ij}^k = 0, \tilde{Z}_i^k = 0$ from (55). Here we will use Lemma 6.5.

For the first equation in (55), by Lemma 6.5, we have

$$\tilde{T}_{ij}^k = \tilde{\theta}_i^k \dot{\gamma}_j + \tilde{\theta}_j^k \dot{\gamma}_i, \quad \tilde{\theta}_i^k = \tilde{T}_{ir}^k \dot{\gamma}^r - \frac{1}{2} \tilde{T}_{rm}^k \dot{\gamma}^r \dot{\gamma}^m \dot{\gamma}_i, \quad (\dot{\gamma}_i := g_{ir}(\dot{\gamma}) \dot{\gamma}^r). \quad (56)$$

For the second equation in (55), we have

$$\tilde{Z}_i^k = \tilde{\lambda}^k \dot{\gamma}_i, \quad \tilde{\lambda}^k = \tilde{Z}_r^k \dot{\gamma}^r. \quad (57)$$

Finally, we can obtain (37) by rewriting (56), $\tilde{S}^k = 0$ and (57) as equations on the tangent bundle TM , using the expressions of $\tilde{T}_{ij}^k, \tilde{Z}_i^k, \tilde{S}^k$ in Lemma 6.4. In the rewriting, we should note that

$$\mathcal{L}_V C_{ij}^k - (\mathcal{L}_{V^c} \ln F) C_{ij}^k = F \mathcal{L}_{V^c} C_{ij}^k, \quad (\mathcal{L}_V g_{ij}) \dot{\gamma}^i \dot{\gamma}^j = F^{-2} V^c(F^2)$$

This completes the proof of Theorem 6.1. Q.E.D.

6.2 Proofs of Theorems 1.1 and 1.2

Using Theorem 6.1, we can complete the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1 : If V is conformal satisfying $\mathcal{L}_{V^c}g_{ij} = 2\rho g_{ij}$ with $\rho = \rho(x)$ being a scalar function on M , then we have

$$0 = \mathcal{L}_{V^c}\delta_j^i = \mathcal{L}_{V^c}(g^{ir}g_{rj}) = (\mathcal{L}_{V^c}g^{ir})g_{rj} + g^{ir}(\mathcal{L}_{V^c}g_{rj}) = (\mathcal{L}_{V^c}g^{ir})g_{rj} + 2\rho\delta_j^i,$$

which gives $\mathcal{L}_{V^c}g^{ij} = -2\rho g^{ij}$. Thus we have (by $\dot{\partial}_r\mathcal{L}_{V^c} = \mathcal{L}_{V^c}\dot{\partial}_r$)

$$\mathcal{L}_{V^c}C_{jk}^i = \mathcal{L}_{V^c}(g^{ir}C_{rjk}) = (\mathcal{L}_{V^c}g^{ir})C_{rjk} + \frac{1}{2}g^{ir}\dot{\partial}_r\mathcal{L}_{V^c}g_{jk} = 0.$$

So the Lie derivative of the mean Cartan torsion I_i along V^c vanishes ($\mathcal{L}_{V^c}I_i = 0$).

Conversely, first assume V is concircular. Then V satisfies (37) in Theorem 6.1. For the first equation of (37), the contraction over the indices j and k gives

$$\mathcal{L}_{V^c}y_i = F^{-2}V^c(F^2)y_i + \frac{2}{3n}F^2\mathcal{L}_{V^c}I_i, \quad (58)$$

where n is the dimension of the Finsler manifold (M, F) . Further assume the Lie derivative of the mean Cartan torsion along V^c vanishes. Then by (58) we have

$$\mathcal{L}_{V^c}y_i = \lambda y_i, \quad (\lambda := F^{-2}V^c(F^2)). \quad (59)$$

Differentiating (59) by y^j we obtain

$$\mathcal{L}_{V^c}g_{ij} = \lambda_{.j}y_i + \lambda g_{ij}, \quad (60)$$

from which we get $\lambda_{.j}y^i = \lambda_{.i}y^j$. Using this and the contraction on both sides by y^j , we immediately have $\lambda_{.i} = 0$ since the zero homogeneity of λ gives $\lambda_{.j}y^j = 0$. So by (60), V is conformal satisfying $\mathcal{L}_{V^c}g_{ij} = \lambda g_{ij}$ for a scalar function $\lambda = \lambda(x)$. Q.E.D.

Proof of Theorem 1.2 : By assumption, V is conformal satisfying $\mathcal{L}_{V^c}g_{ij} = 2\rho g_{ij}$ with $\rho = \rho(x)$ being a scalar function on M . Then we have

$$\mathcal{L}_{V^c}C_{ij}^k = 0, \quad \mathcal{L}_{V^c}y_i = 2\rho y_i, \quad (\mathcal{L}_{V^c}y_i)_{|j} = 2\rho_j y_i, \quad (\rho_i := \rho_{x^i}), \quad (61)$$

$$V^c(F^2) = 2\rho F^2, \quad [V^c(F^2)]_{|i} = 2F^2\rho_i, \quad [V^c(F^2)]_{|i|j} = 2F^2\rho_{i|j}. \quad (62)$$

By $\mathcal{L}_{V^c}g_{ij} = 2\rho g_{ij}$, we have (by Lemma 4.1)

$$V_{i|0} + V_{0|i} = 2\rho y_i, \quad V_{i|0|0} + V_{0|i|0} = 2\rho_0 y_i, \quad V_{0|0} = \rho F^2. \quad (63)$$

Then by a Ricci identity of Cartan connection and (63) we have

$$V_{0|i|0} = V_{0|0|i} + V_{0..m}R^m_i = V_{0|0|i} + V_m R^m_i = F^2\rho_i + V_m R^m_i. \quad (64)$$

Plugging (64) into the second formula of (63) we obtain

$$V_{i|0|0} + V_m R^m_i = 2\rho_0 y_i - F^2\rho_i, \quad V^i_{|0|0} + V^m R^i_m = 2\rho_0 y^i - F^2\rho^i \quad (65)$$

From (65), Lemmas 4.1 and 4.2, we obtain (note that $A_{j0}^k = B_{j0}^k$)

$$A_{00}^k = 2\rho_0 y^k - F^2\rho^k, \quad A_{00|0}^k = 2\rho_{0|0} y^k - F^2\rho^k_{|0}, \quad (66)$$

$$A_{j0}^k = \frac{1}{2}A_{00..j}^k = \rho_j y^k + \rho_0 \delta_j^k - y_j \rho^k + F^2 \rho^r C_{jr}^k. \quad (67)$$

Now by Theorem 6.1, we see that V is concircular iff. (37) holds. So we only need to simplify (37) with the help of (61), (62), (66) and (67). By (61) and (62), we see the first equation of (37) automatically holds. From (62) and (66), the second equation of (37) is reduced to

$$\rho_{0|0}y^k = F^2\rho^k_{|0}. \quad (68)$$

By (61), (62), (66) and (67), the third equation of (37) is reduced to $\rho^r C_{ir}^k = 0$, since we have

$$Z_i^k = 2F^2(F^2\rho^r C_{ir}^k - 3\rho^k y_i), \quad \lambda^k = -6F^4\rho^k.$$

By (68) we have

$$\rho^i_{|0} = \tau y^i, \quad \rho_{i|0} = \tau y_i, \quad (\tau := F^{-2}\rho_{0|0}). \quad (69)$$

Differentiating (69) by y^j we obtain (by a Ricci identity of Berwald connection)

$$\rho_{i;j} = \tau g_{ij} + \tau_{;j} y_i \quad (\Longleftrightarrow \quad \rho_{i|j} - \rho_r C_{ij|0}^r = \tau g_{ij} + \tau_{;j} y_i), \quad (70)$$

from which we again get $\tau_{;j} y_i = \tau_{;i} y_j$. Thus we have $\tau_{;i} = 0$, which means that τ is a scalar function on M . From (70) we have $\rho_{i;j} = \tau g_{ij}$, or equivalently $\rho_{i|j} = \tau g_{ij}$ since $\rho^r C_{ir}^k = 0$ and (69) imply $\rho_r C_{ij|0}^r = 0$. Now we have obtained (2). Q.E.D.

6.3 Proof of Theorem 1.3

We first show the following lemma which is needed in the proof of Theorem 1.3 (i).

Lemma 6.6 *Let \tilde{F} and F be two conformally related Finsler metrics on a same manifold M with $\tilde{F} = u^{-1}F$, and $\gamma = \gamma(s)$ be a curve with $F(\dot{\gamma}(s)) = 1$. Then we have*

$$\tilde{D}_{\dot{\gamma}}^* \tilde{D}_{\dot{\gamma}}^* \dot{\gamma} - 3 \frac{\tilde{g}_{\dot{\gamma}}(\dot{\gamma}, \tilde{D}_{\dot{\gamma}}^* \dot{\gamma})}{\tilde{F}^2(\dot{\gamma})} \tilde{D}_{\dot{\gamma}}^* \dot{\gamma} = D_{\dot{\gamma}}^* D_{\dot{\gamma}}^* \dot{\gamma} + \frac{1}{u} D_{\dot{\gamma}}^* U + \lambda \dot{\gamma}, \quad (71)$$

where $U = u^i \partial_i$ is a vector field along γ defined by $u^i := g^{ir}(\dot{\gamma}) u_r$ with $u_r := u_{x^r}$, and $\lambda = \lambda(s)$ is a function along γ .

Proof : Since $\tilde{F} = u^{-1}F$, a direct computation from (5) shows that

$$\tilde{G}^i = G^i - \frac{1}{u} u_0 y^i + \frac{1}{2u} F^2 u^i, \quad \tilde{G}_j^i = G_j^i - \frac{1}{u} (u_j y^i + u_0 \delta_j^i - y_j u^i + F^2 C_{jr}^i u^r). \quad (72)$$

By the first formula of (72) we have

$$(\tilde{D}_{\dot{\gamma}}^* \dot{\gamma})^k = \dot{\gamma}^k + 2\tilde{G}^k = (D_{\dot{\gamma}}^* \dot{\gamma})^k - \frac{2}{u} g_{\dot{\gamma}}(\dot{\gamma}, U) \dot{\gamma}^k + \frac{1}{u} u^k. \quad (73)$$

Then by (73), we first have

$$\tilde{D}_{\dot{\gamma}}^* \tilde{D}_{\dot{\gamma}}^* \dot{\gamma} = \tilde{D}_{\dot{\gamma}}^* D_{\dot{\gamma}}^* \dot{\gamma} - \frac{d}{ds} \left[\frac{2}{u} g_{\dot{\gamma}}(\dot{\gamma}, U) \right] \dot{\gamma} - \frac{2}{u} g_{\dot{\gamma}}(\dot{\gamma}, U) \tilde{D}_{\dot{\gamma}}^* \dot{\gamma} + \frac{d}{ds} \left(\frac{1}{u} \right) U + \frac{1}{u} \tilde{D}_{\dot{\gamma}}^* U. \quad (74)$$

We respectively have

$$\begin{aligned} (\tilde{D}_{\dot{\gamma}}^* D_{\dot{\gamma}}^* \dot{\gamma})^k &= \frac{d}{ds} (D_{\dot{\gamma}}^* \dot{\gamma})^k + \dot{\gamma}^j (D_{\dot{\gamma}}^* \dot{\gamma})^i \tilde{\Gamma}_{ij}^{*k} = \frac{d}{ds} (D_{\dot{\gamma}}^* \dot{\gamma})^k + \dot{\gamma}^j (D_{\dot{\gamma}}^* \dot{\gamma})^i (*\tilde{\Gamma}_{ij}^k + \tilde{C}_{ir}^k \tilde{Y}_j^r) \\ &= \frac{d}{ds} (D_{\dot{\gamma}}^* \dot{\gamma})^k + (D_{\dot{\gamma}}^* \dot{\gamma})^i [\tilde{G}_i^k + \tilde{C}_{ir}^k (\tilde{D}_{\dot{\gamma}}^* \dot{\gamma})^r], \end{aligned} \quad (75)$$

$$(\tilde{D}_{\dot{\gamma}}^* U)^k = \frac{d}{ds} u^k + u^i [\tilde{G}_i^k + \tilde{C}_{ir}^k (\tilde{D}_{\dot{\gamma}}^* \dot{\gamma})^r], \quad (76)$$

By $\tilde{C}_{jk}^i = C_{jk}^i$ (conformally invariant), the second formula of (72) and (73), we can rewrite (75) and (76) as follows:

$$\tilde{D}_\gamma^* D_\gamma^* \dot{\gamma} = D_\gamma^* D_\gamma^* \dot{\gamma} - \frac{1}{u} \{g_\gamma(U, D_\gamma^* \dot{\gamma}) \dot{\gamma} + g_\gamma(U, \dot{\gamma}) D_\gamma^* \dot{\gamma}\}, \quad (77)$$

$$\tilde{D}_\gamma^* U = D_\gamma^* U - \frac{1}{u} g_\gamma(U, U) \dot{\gamma}. \quad (78)$$

Plugging (73), (77) and (78) into (74), we obtain

$$\tilde{D}_\gamma^* \tilde{D}_\gamma^* \dot{\gamma} = D_\gamma^* D_\gamma^* \dot{\gamma} + \frac{1}{u} D_\gamma^* U - \frac{3}{u} g_\gamma(\dot{\gamma}, U) (\tilde{D}_\gamma^* \dot{\gamma} + \frac{1}{u} U) + \lambda_1 \dot{\gamma}, \quad (79)$$

where $\lambda_1 = \lambda_1(s)$ is a function along γ . By $\tilde{F} = u^{-1}F$ and (73), it is easy to see that

$$-3 \frac{\tilde{g}_\gamma(\dot{\gamma}, \tilde{D}_\gamma^* \dot{\gamma})}{\tilde{F}^2(\dot{\gamma})} \tilde{D}_\gamma^* \dot{\gamma} = \frac{3}{u} g_\gamma(\dot{\gamma}, U) (\tilde{D}_\gamma^* \dot{\gamma} + \frac{1}{u} U) + \lambda_2 \dot{\gamma}, \quad (80)$$

where $\lambda_2 = \lambda_2(s)$ is a function along γ . By (79) and (80), we immediately obtain (71). This completes the proof. Q.E.D.

Now we can get started with the proof of Theorem 1.3 (i). Let \tilde{F} be conformally related to F satisfying $\tilde{F} = u^{-1}F$ on a same manifold M .

Assume u satisfies (3). Let $\gamma = \gamma(s)$ be an arbitrary geodesic circle of (M, F) with $F(\dot{\gamma}(s)) = 1$. Then by the definition of a geodesic circle, we see $D_\gamma^* D_\gamma^* \dot{\gamma}$ is parallel to $\dot{\gamma}$. By (7) and then by (3), we have

$$(D_\gamma^* U)^k = \dot{\gamma}^j u_{|j}^k = \dot{\gamma}^j (u_{|j}^k + u^k |_r Y_j^r) = \dot{\gamma}^j u_{|j}^k + u^m C_{mr}^k (D_\gamma^* \dot{\gamma})^r = \lambda \dot{\gamma}^k.$$

By Lemma 6.6 we have (71). Now it is easy to see that (71) implies that the following vector

$$\tilde{D}_\gamma^* \tilde{D}_\gamma^* \dot{\gamma} - 3 \frac{\tilde{g}_\gamma(\dot{\gamma}, \tilde{D}_\gamma^* \dot{\gamma})}{\tilde{F}^2(\dot{\gamma})} \tilde{D}_\gamma^* \dot{\gamma}$$

is parallel to $\dot{\gamma}$. Thus by Proposition 3.7, the curve γ is also a geodesic circle (as points set) of (M, \tilde{F}) . Similarly, a geodesic circle of \tilde{F} is also a geodesic circle of F . This means that \tilde{F} and F are concircular.

Conversely, suppose that \tilde{F} and F are concircular. Then an arbitrary geodesic circle γ of (M, F) is also a geodesic circle of (M, \tilde{F}) . So by Proposition 3.7 and (71), we see that $D_\gamma^* U$ is parallel to $\dot{\gamma}$, which shows that

$$((D_\gamma^* U)^k = \dot{\gamma}^j u_{|j}^k) \Rightarrow \dot{\gamma}^j u_{|j}^k + u^m C_{mr}^k (D_\gamma^* \dot{\gamma})^r = \lambda \dot{\gamma}^k, \quad (81)$$

where $\lambda = \lambda(\gamma, \dot{\gamma})$ is a scalar function along γ , and actually the contraction of (81) by $\dot{\gamma}_k := g_{kr}(\dot{\gamma}) \dot{\gamma}^r$ gives $\lambda = \dot{\gamma}^i \dot{\gamma}^j u_{i|j}$. Let $w := \dot{\gamma}$ and W^\perp be the $(n-1)$ -dimensional space perpendicular to w with respect to the inner product g_w . Then for fixed $\dot{\gamma}$, by Proposition 3.2, we see that (81) is a polynomial equation of the variable $X := D_\gamma^* \dot{\gamma} \in W^\perp$. Thus (81) is equivalent to

$$\dot{\gamma}^j u_{|j}^k = \lambda \dot{\gamma}^k, \quad u^m C_{mr}^k (D_\gamma^* \dot{\gamma})^r = 0. \quad (82)$$

We can write (82) as equations on the tangent bundle TM as follows:

$$u_{|0}^k = \lambda y^k, \quad u^m C_{mr}^k = \tau^k y_r. \quad (83)$$

The first equation in (83) is similar to (69). So $\lambda = \lambda(x)$ is a scalar function on M , and then $u_{i;j} = \lambda g_{ij}$. For the second equation of (83), the contraction by y^r immediately gives $\tau^k = 0$ and thus $u^m C_{mr}^k = 0$. Now we have proved (3).

Before the proof of Theorem 1.3 (ii), we first give a brief introduction for some basic points needed here. It is known that if two sprays \tilde{G}^i and G^i satisfy $\tilde{G}^i = G^i + H^i$, then their Riemann curvature tensors \tilde{R}^i_k and R^i_k are related by

$$\tilde{R}^i_k = R^i_k + 2H^i_{;k} - y^m H^i_{;m \cdot k} + 2H^m H^i_{\cdot m \cdot k} - H^i_{\cdot m} H^m_{\cdot k}, \quad (84)$$

where the symbol $;$ denotes the horizontal covariant derivative of Berwald connection of G^i . A Finsler metric F is said to be of scalar (resp. isotropic) flag curvature, if the Riemann curvature satisfies

$$R^i_k = K(F^2 \delta_k^i - y^i y_k), \quad (85)$$

where $K = K(x, y)$ is a scalar function on TM (resp. $K = K(x)$ is a scalar function on M). If K in (85) is a constant, then F is called of constant flag curvature. A Finsler metric F is said to be an Einstein metric, if the Ricci curvature is of isotropic Ricci scalar in the following form,

$$Ric = (n-1)KF^2, \quad (86)$$

where $K = K(x)$ is a scalar function on M .

Now we show the proof. Since $\tilde{F} = u^{-1}F$, the sprays \tilde{G}^i and G^i are related by (72) with H^i being given by

$$H^i = -\frac{1}{u}u_0 y^i + \frac{1}{2u}F^2 u^i. \quad (87)$$

Plugging (87) into (84), we obtain by a direct computation

$$\begin{aligned} \tilde{R}^i_k &= R^i_k + \frac{uu_{0;0} - (u_m u^m)F^2}{u^2} \delta_k^i + \frac{1}{u}F^2 u^i_{;k} + \frac{u_m u^m}{u^2} y^i y_k - \frac{1}{u}(y^i u_{k;0} + y_k u^i_{;0}) \\ &\quad - \frac{u^m u^r}{u^2} F^2 (y^i C_{kmr} + y_k C^i_{mr}) + \frac{1}{u^2} F^2 (uu^r_{;0} - 3u_0 u^r) C^i_{kr} + \frac{1}{u} F^2 u^r C^i_{kr;0} \\ &\quad + \frac{u^r u^m}{u^2} F^4 (C^i_{pr} C^p_{km} - C^i_{mr \cdot k}). \end{aligned} \quad (88)$$

Then by (88), the Ricci curvatures $\widetilde{Ric} := \tilde{R}^m_m$ and $Ric := R^m_m$ are related by

$$\begin{aligned} \widetilde{Ric} &= Ric + \frac{n-2}{u} u_{0;0} + \frac{1}{u^2} [uu^m_{;m} - (n-1)u^m u_m + uI^r u_{r;0} + u^r(uI_{r;0} - 3u_0 I_r)] F^2 \\ &\quad - \frac{1}{u^2} u^r u^m (C^i_{jm} C^j_{ir} - 2I^i C_{imr} + I_{m \cdot r}) F^4 \end{aligned} \quad (89)$$

Now suppose \tilde{F} and F are concircular. Then by Theorem 1.3 (i), we have (3). Plugging (3) into (88) and (89), we respectively have

$$\tilde{R}^i_k = R^i_k + u^{-2}(2\lambda u - u_m u^m)(F^2 \delta_k^i - y^i y_k), \quad (90)$$

$$\widetilde{Ric} = Ric + (n-1)u^{-2}(2\lambda u - u_m u^m)F^2. \quad (91)$$

If F is of scalar (resp. isotropic) flag curvature satisfying (85), or an Einstein metric satisfying (86), then plugging (85) into (90), and (86) into (91), respectively we obtain

$$\tilde{R}^i_k = (Ku^2 + 2\lambda u - u_m u^m)(\tilde{F}^2 \delta_k^i - y^i \tilde{y}_k), \quad (92)$$

$$\widetilde{Ric} = (n-1)(Ku^2 + 2\lambda u - u_m u^m)\tilde{F}^2. \quad (93)$$

Note that we have $u_{,k}^i = 0$ from the second equation in (3). Now it follows from (92) that \tilde{F} is of scalar (resp. isotropic) flag curvature \tilde{K} given by (4), or from (93) that \tilde{F} is an Einstein metric with the Ricci scalar \tilde{K} given by (4). Q.E.D.

Remark 6.7 In Theorem 1.3 (i), if F is locally Euclidean, then we can solve (3) in a local coordinate such that \tilde{F} is locally expressed as $\tilde{F} = u^{-1}|y|$. So $u_{i|j} = \lambda g_{ij}$ is equivalent to $u_{x^i x^j} = \lambda \delta_{ij}$. By integrability, we see λ is a constant, and thus we obtain

$$\tilde{F} = (a|x|^2 + \langle b, x \rangle + c)^{-1}|y|, \quad (a := \lambda/2), \quad (94)$$

where a, c are constant numbers and b is a constant n -vector such that $u > 0$.

For convenience, suppose (94) is defined on the whole R^n . Let $\gamma = \xi s + \tau$ be a geodesic in the Euclidean space (R^n, F) , where ξ, τ are n -vectors satisfying $|\xi| = 1$. Let t be the arc-length of γ with respect to \tilde{F} . Then a direct computation from (72) gives

$$\tilde{D}_{\gamma'(t)}^* \gamma'(t) = u[2a\tau + b - \langle 2a\tau + b, \xi \rangle \xi].$$

So γ is also a geodesic of \tilde{F} iff. $2a\tau + b$ is tangent to γ . Otherwise, γ is a circle of \tilde{F} .

Let $\gamma = \gamma(s)$ be a circle of F . Then by Example 3.6, γ is written as

$$\gamma = \xi \cos ks + \eta \sin ks + \tau, \quad |\xi| = |\eta| = 1/k.$$

Similarly, a direct computation gives

$$\tilde{D}_{\gamma'(t)}^* \gamma'(t) = u[(A \cos ks - k^2 \langle B, \xi \rangle) \xi + (A \sin ks - k^2 \langle B, \eta \rangle) \eta + B], \quad (95)$$

where t is the arc-length of γ with respect to \tilde{F} , and A, B are defined by

$$A := a - k^2(\langle b, \tau \rangle + a|\tau|^2 + c), \quad B := b + 2a\tau.$$

By (95), we easily obtain

$$|\tilde{D}_{\gamma'(t)}^* \gamma'(t)|_{\tilde{g}_{\gamma'(t)}}^2 = -(\langle B, \xi \rangle^2 + \langle B, \eta \rangle^2)k^2 + |B|^2 + \frac{A^2}{k^2}.$$

Thus we can determine the conditions for γ to be a geodesic or a circle of \tilde{F} .

7 Some examples

In this section, we give some examples to show that concircular vector fields might not be conformal and conformal vector fields might not be concircular.

Example 7.1 Let $F = \alpha + \beta$ be an n -dimensional Randers metric. By the first equation in (37), we can prove that if $n \geq 3$, then any concircular vector field of F must be conformal. While in dimension $n = 2$, there exist non-conformal concircular vector fields, which will be exemplified as follows.

Define a two-dimensional Minkowskin Randers metric $F = \alpha + \beta$ by

$$\alpha := \sqrt{(y^1)^2 + (y^2)^2}, \quad \beta := by^1, \quad (a \text{ constant } b \text{ with } 0 < b < 1),$$

and a vector field $V = (V^1, V^2)$ by

$$V^1 := qx^2 + \eta^1, \quad V^2 := -qx^1 + \eta^2,$$

where q is a non-zero constant and $\eta = (\eta^1, \eta^2)$ is a constant vector. It can be easily checked that

$$V^c(\alpha^2) = 0, \quad V^c(\beta) = b q y^2 \neq 0,$$

which implies that V is not conformal in F (cf. [20, 21]). On the other hand, a direct verification shows that V satisfies (37), and thus V is concircular in F by Theorem 6.1.

Example 7.2 Let $F = e^{\sigma(x)/2}|y|$ be an $n(\geq 3)$ -dimensional conformally flat Riemann metric and V be a conformal vector field of F with the conformal factor $\rho = \rho(x)$. Then

$$V^i = -2(\lambda + \langle d, x \rangle)x^i + |x|^2 d^i + q_r^i x^r + \eta^i, \quad \rho = -2(\lambda + \langle d, x \rangle) + \frac{1}{2}V(\sigma),$$

where λ is a constant number, d, η are constant vectors and (q_i^j) is skew-symmetric (cf. [18, 19, 20, 21]). If F is of constant sectional curvature μ ($\sigma = \ln 4/(1 + \mu|x|^2)^2$), then V is also concircular by (2) (cf. [16]). Taking $\sigma = |x|^2$ (or many other functions), we can check that V is non-concircular by (2).

Example 7.3 Define a projectively flat Randers metric $F = \alpha + \beta$ and a vector field V by

$$\alpha := \frac{2}{1 + \mu|x|^2}|y|, \quad \beta := \frac{1}{\lambda(1 - \mu|x|^2) + \langle d, x \rangle} \left\{ \langle d, y \rangle - \frac{2\mu(2\lambda + \langle d, x \rangle)\langle x, y \rangle}{1 + \mu|x|^2} \right\},$$

$$V^i := -2(\lambda + \langle d, x \rangle)x^i + |x|^2 d^i,$$

where the constant λ and the constant vector $d = (d^i) \neq 0$ satisfy $|d|^2 + 4\mu\lambda^2 = 0$. It has been verified in [20, 21] that V is a non-homothetic conformal vector field in F with the conformal factor ρ given by

$$\rho = -\frac{2[\lambda(1 - \mu|x|^2) + \langle d, x \rangle]}{1 + \mu|x|^2}.$$

It can be checked directly that ρ does not satisfy (2), and so V is not concircular in F by Theorem 1.2.

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Zhongmin Shen
 Department of Mathematical Sciences
 Indiana University Purdue University Indianapolis (IUPUI)
 402 N. Blackford Street
 Indianapolis, IN 46202-3216, USA
 zshen@math.iupui.edu

Guojun Yang
 Department of Mathematics
 Sichuan University
 Chengdu 610064, P. R. China
 yangguojun@scu.edu.cn